MATH 1650 RATIONAL FUNCTION INEQUALITIES

To solve an inequality involving a rational function, we use a Sign Diagram.

EXAMPLE: Solve $\dfrac{5-x}{x^2-x-2} \geq 0$. Write your answer using interval notation.

Letting $f(x) = \dfrac{5-x}{x^2-x-2}$, we use a Sign Diagram to solve $f(x) \geq 0$.

We first find values excluded from the domain of f. To solve $x^2 - x - 2 = 0$, we factor: $(x+1)(x-2) = 0$.

We get two excluded values: $x = -1$ and $x = 2$.

Next, we find the zeros of f by solving $f(x) = \dfrac{5-x}{x^2-x-2} = 0$. We get $5 - x = 0$ so $x = 5$.

Our Sign Diagram is:

$$\begin{array}{cccccccc} (+) & ? & (-) & ? & (+) & 0 & (-) & f(x) \\ \longleftarrow & -1 & \text{---} & 2 & \text{---} & 5 & \longrightarrow & x \end{array}$$

Hence, $f(x) \geq 0$ on $(-\infty, -1) \cup (2, 5]$.

EXAMPLE: Solve $\dfrac{x+1}{2x-5} \leq 1$. Write your answer using interval notation.

We rewrite the inequality so as to compare with 0 and get a common denominator:

$$\begin{aligned} \frac{x+1}{2x-5} - 1 &\leq 0 \\ \frac{x+1}{2x-5} - \frac{1}{1} &\leq 0 \\ \frac{x+1}{2x-5} - \frac{2x-5}{2x-5} &\leq 0 \\ \frac{(x+1)-(2x-5)}{2x-5} &\leq 0 \\ \frac{-x+6}{2x-5} &\leq 0 \end{aligned}$$

We let $f(x) = \dfrac{-x+6}{2x-5}$ and make a Sign Diagram to solve $f(x) \leq 0$.

To find the values excluded from the domain of f, we solve $2x - 5 = 0$ so $x = \frac{5}{2}$.

To find the zeros of f we solve $f(x) = \dfrac{-x+6}{2x-5} = 0$. We get $-x + 6 = 0$ so $x = 6$.

Our Sign Diagram for $f(x)$ is below.

$$\begin{array}{cccccc} (-) & ? & (+) & 0 & (-) & f(x) \\ \longleftarrow & \frac{5}{2} & \text{---} & 6 & \longrightarrow & x \end{array}$$

Solving $f(x) \leq 0$, we get $\left(-\infty, \frac{5}{2}\right) \cup [6, \infty)$

MATH 1650 APPLICATIONS OF RATIONAL FUNCTIONS

EXAMPLE: The cost $C(x)$ in millions of dollars to seize $x\,\%$ of the illegal drug 'Shadow' is given by:

$$C(x) = \frac{300x}{100-x}, \ 0 \le x < 100$$

- Find and interpret $C(20)$.

  $C(20) = \dfrac{300(20)}{100-20} = 75$. It costs \$75 million to seize 20 % of Shadow.

- How much Shadow could be seized if \$100 million were spent on its removal?

  We solve $C(x) = 100$:

$$\begin{aligned} C(x) &= 100 \\ \frac{300x}{100-x} &= 100 \\ 300x &= 100(100-x) \\ 300x &= 10000 - 100x \\ 400x &= 10000 \\ x &= 25 \end{aligned}$$

  For \$100 million, we could seize 25 % of Shadow.

- Find and interpret the vertical asymptote to the graph of $y = C(x)$.

  The value excluded from the domain of $C(x) = \dfrac{300x}{100-x}$ is $x = 100$.

  Since there is no cancellation, $x = 100$ is a vertical asymptote to the graph of $y = C(x)$.

  Moreover, as $x \to 100^-$, $C(x) \to \infty$. Hence, we'll never be able to seize all (100 %) of Shadow since the cost becomes unbounded (more and more prohibitive.)

DEFINITION: If $C(x)$ is the cost of producing x items, the **average cost per item**, denoted $\overline{C}(x)$ is given by:

$$\overline{C}(x) = \frac{C(x)}{x}, \quad x > 0$$

EXAMPLE: The cost, in dollars, to produce x "I'd rather be a Sasquatch" T-Shirts is $C(x) = 2x + 26$, $x \geq 0$.

- Find an expression for $\overline{C}(x)$.

  $\overline{C}(x) = \dfrac{C(x)}{x} = \dfrac{2x + 26}{x}$ for $x > 0$.

- Find and interpret $\overline{C}(20)$.

  $\overline{C}(20) = \dfrac{2(20) + 26}{20} = \dfrac{66}{20} = 3.3$. When 20 T-shirts are made, the average cost is \$3.30 per T-Shirt.

- Solve and interpret $\overline{C}(x) = 3$.

$$\begin{aligned} \overline{C}(x) &= 3 \\ \frac{2x+26}{x} &= 3 \\ 2x + 26 &= 3x \\ x &= 26 \end{aligned}$$

  This means that in order for the average cost to reach \$3 per T-Shirt, we must make 26 T-Shirts.

- Find and interpret the horizontal asymptote of the graph of $y = \overline{C}(x)$.

  As $x \to \infty$, $\overline{C}(x) = \dfrac{2x + 26}{x} \approx \dfrac{2x}{x} = 2$.

  As more and more T-Shirts are made, the average cost gets closer to \$2 per T-Shirt.

  NOTE: You may recall that \$2 per T-Shirt was the **variable** cost of this venture back in Chapter 1.

  Rewriting $\overline{C}(x)$ we find:

$$\overline{C}(x) = \frac{2x + 26}{x} = \frac{2x}{x} + \frac{26}{x} = 2 + \frac{26}{x}$$

  The term $\dfrac{26}{x}$ represents the fixed cost, \$26 divided by the number of T-Shirts made, x.

  In other words, as $x \to \infty$, the fixed cost gets phased out so the average cost approaches the variable cost!